Digital Control Systems
MAE/ECEN 5473

Controller and observer design

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Control design for SS DTS

$$x(k+1) = Gx(k) + Hu(k), \quad y(k) = Cx(k),\, x(k) \in \mathbb{R}^n$$

Before designing any controller, check if the system is controllable.

Controllability (completely state controllable)

There exists a piecewise constant control $u(k)$ such that from any initial condition of $x(k)$, $x(0)$, the state $x(k)$ can be transferred to any desired state $x_f \in \mathbb{R}^n$ in a finite time.

- Finite time:
- Control signal $u(k)$

Derivation of controllability condition

Examples

Determine controllability: Diagonal form

Determine controllability: General form

Example

Observability

$$x(k+1) = Gx(k) + Hu(k), \quad y(k) = Cx(k) + Du(k),\ x(k) \in \mathbb{R}^n$$

- Dual concept to controllability, required for observer design

(state) observability

Every initial condition $x(0)$ can be determined by observations $y(k)$ and control inputs $u(k)$ over a finite number of sampling instants.

Duality

Example $\frac{Y(z)}{U(z)} = \frac{z+0.8}{z^2+1.3z+0.4}$

Observability and controllability of a PTF

Sampling effect on controllability and observability

Example

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = [1 \ \ 0]x.$$

Controller design: pole placement (PP)

Consider the open-loop system $x(k+1) = Gx(k) + Hu(k)$, $x(k) \in \mathbb{R}^n$.

- Assume a full state feedback: $u(k) = -Kx(k)$, $K \in \mathbb{R}^{1\times n}$: control gain to be designed.
- Objective: choose K such that the closed-loop poles are at desired locations $z_1 = \mu_1, \cdots, z_n = \mu_n$, where μ_i's are given.
- Closed-loop system: $x(k+1) = Gx(k) + Hu(k) =$
- Closed-loop poles are eigenvalues of
- Necessary & sufficient condition for PP: the system is controllable.

Solutions to PP

$$PP \Leftrightarrow |zI - (G - H \underbrace{K}_{unknown})| = \underbrace{(z - \mu_1)(z - \mu_2) \cdots (z - \mu_n)}_{unknown}$$

Solution 1 (Good for low order systems)

1. Let $K =$
2. Compute
3. Match
4. Use

Example

$$x(k+1) = \underbrace{\begin{pmatrix} 0 & 1 \\ -0.16 & -1 \end{pmatrix}}_{G} x(k) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{H} u(k). \text{ Find } u(k) = -Kx(k)$$

such that the closed-loop poles are at $z_{1,2} = 0.5 \pm 0.5j$.

Solution 2: formula

- Original characteristic equation:
  $|zI - G| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$
  Desired characteristic equation:
  $(z - \mu_1) \cdots (z - \mu_n) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + a_n$
- $K =$

- special case: if (G, H) is already in CCF,

Example

$$x(k+1) = \underbrace{\begin{pmatrix} 0 & 1 \\ -0.16 & -1 \end{pmatrix}}_{G} x(k) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{H} u(k). \text{ Find } u(k) = -Kx(k)$$

such that the closed-loop poles are at $z_{1,2} = 0.5 \pm 0.5j$.

Solution 3: Ackermann formula

Example

$$x(k+1) = \underbrace{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}_{G} x(k) + \underbrace{\begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix}}_{H} u(k). \text{ Find } u(k) = -Kx(k)$$

such that the closed-loop poles are at $z_{1,2} = 0, 0$.

Final comments

- Deadbeat control
  - All the poles are placed at 0: unique to DTS
  - $x(k)$ converges to 0 in n steps!!
  - Price paid: large control effort
- Where to place the closed-loop poles
  - Inside the unit circle
  - Examine a few sets of closed-loop poles: tradeoff between convergence speed & sensitivity to noise and disturbance.

Observer design

- Objective: reconstructs $x(k)$ based on $y(k)$ and $u(k)$

- Motivating example

Observer structure

$$x(k+1) = Gx(k) + Hu(k),\ y(k) = Cx(k) + Du(k)$$

- Basic requirement: the system must be observable.
- General structure of an observer:
  $\tilde{x}(k+1) = G\tilde{x}(k) + Hu(k) + K_e[y(k) - (C\tilde{x}(k) + Du(k))]$

- Design K_e such that $|\tilde{x}(k) - x(k)| \to 0$ as $k \to \infty$.

Condition on K_e

- Define the error $e(k) = \tilde{x}(k) - x(k)$ and derive its dynamics.

- Objective $e(k) \to 0$, $\forall x(0) \Leftrightarrow$
  $\Leftrightarrow$

- Because (G, C) is observable,

Design K_e

Equivalence to a PP problem

Solution 1: direct computation

Given desired poles of $G - K_e C$ as $\mu_1, \cdots, \mu_n$

Solution 2

Solution 3

Example

$$x(k+1) = \underbrace{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}_{G} x(k) + \underbrace{\begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix}}_{H} u(k),\ y = [1 \ \ 0]x(k).$$ Design an observer such that the desired observer poles are at $0, 0$.

PP + observer design

$$x(k+1) = Gx(k) + Hu(k), \; y = Cx(k).$$

Design $u = -K\tilde{x}(k)$ where K is designed using PP and $\tilde{x}(k)$ is the estimate of $x(k)$ from an observer.

- Closed-loop dynamics

- Eigenvalues

Separation principle

- Design a stabilizing control K assuming
- Design a stable observer to estimate the full state $x(k)$ by $\tilde{x}(k)$
- Implement
- The resulting closed-loop system

1. Stability
2. Observer convergence

Servo control: integral control

- Control objective: regulate the system output to track a reference signal $r(k)$
- When $r(k)$ is a step function, an integral control is typically needed to track $r(k)$.

dynamics: $x(k+1) = Gx(k) + Hu(k)$, $y = Cx(k)$
integral control: $v(k) = v(k-1) + [r(k) - y(k)]$

implemented control: $u(k) = -K_2 x(k) + K_1 v(k)$

- Design K_1 and K_2 such that $y(k)$ tracks a constant $r(k)$ and the closed-loop system is g.a.s.

Main idea

- Augment the state $x(k)$ with $v(k)$, i.e.,
- Derive $\xi(k)$ dynamics.

- The eigenvalues of

At the steady state

Design procedure

1. Design
2. Solve for K_1 and K_2 based on

- If not all states are available, design an observer to estimate $x(k)$ and replace the control with $u(k) = -K_2\tilde{x}(k) + K_1 v(k)$.

Overall block diagram

Example

$$x(k+1) = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -0.12 & -0.01 & 1 \end{pmatrix}}_{G} x(k) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{H} u(k),$$

$$y = [0.5 \;\; 1 \;\; 0] x(k).$$

Design $u(k) = -K_2 x(k) + K_1 v(k)$ for y to track a constant r.