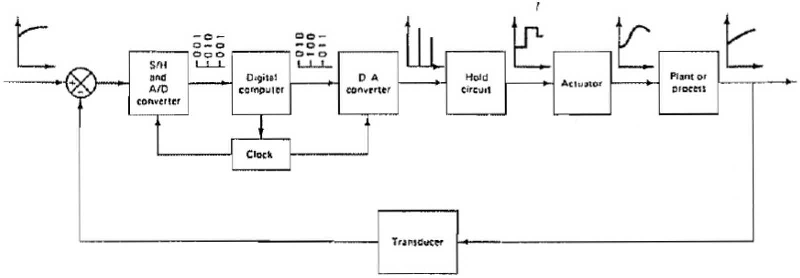
Digital Control Systems
MAE/ECEN 5473

Z Transform

Oklahoma State University

August 14, 2023

DCS overview



Types of signals

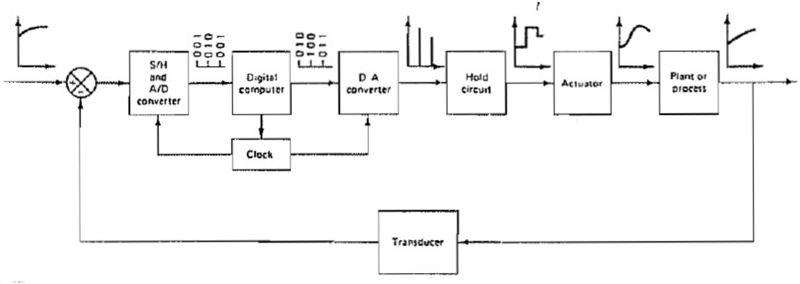
A signal $x(t)$ has the following aspects:

- Time: defined over all the time interval (CT) vs. only at specific time instants (DT)
- Magnitude: real value vs. quantized value
- continuity: whether the signal is continuous over time (not main focus here)

	Real	Quantized
CT	Analog (A)	? (B)
DT	Sampled (C)	Digital (D)

• The distinction between continuous-time vs. analog, discrete-time vs. digital is not that significant, i.e., words used interchangeably.

Identify different types of signals



Expand S/H

Modeling the sample and hold proces: Motivation

- If A/D happens instantaneously and sampling is extremely fast, then digital signals would approach the analog signals.
- The values are sampled and held at specific intervals, causing the system performance change w.r.t. changes in sampling rate.
- Must develop a mathematical model to represent this sample-and-hold process (S/H)
- Once the model for S/H is known, we can start analyzing and designing controls.

Modeling the sampler: the Z transform

- How to represent the sampled signal? Applying Laplace transform is difficult/unwieldy.
- Develop a tool called the z transform that is applicable to sampled signals.
- Sampled signals (discrete-time signals): $x(t)$ sampled at time 0, T, $2T$, ..., where T is the sampling period, represented as $x(kT)$ or sometimes $x(k)$, $k = 0, 1, \cdots,$. (Note the difference $x(kT)$-sampled signal and $x(k)$-just a sequence of numbers.)
- Z transform applies to $x(t)$, $x(kT)$, and $x(k)$.

Z-transform

- Most common: One-sided z transform (assumes $x(t) = 0, t < 0$ or $x(k) = 0, k < 0$)
  - Continuous time signal $x(t)$, $t \geq 0$ or a sampled sequence $x(kT)$, $k \geq 0$, T sampling period

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

  - A sequence of numbers $x(k)$, $k \geq 0$,

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k}$$

- z is complex variable.

Two-sided z transform

Two-sided z transform (assumes $x(t) \neq 0, t < 0$ or $x(k) \neq 0, k < 0$)

- Continuous time signal $x(t)$, or a sampled sequence $x(kT)$

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k}$$

- A sequence of numbers $x(k)$

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

Calculating z transforms

- Given $x(t)$, $X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots$ (or vice versa)
- Z transforms of elementary functions

Example

Z transform of step input.

Example: Z transform of Ramp input

Comments

Note on convergence

In calculating the Z transform of step response, note that the series converges if $|z| > 1$. In finding z transforms, z is a dummy variable. As long as there is a region of z for which $X(z)$ is converging, it is fine. We don't have to specify it.

Note on validity

The obtained $X(z)$ is valid throughout the z plane except at poles of $X(z)$.

Table of z tranforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

Example

Table of z transforms also allows conversion from Laplace transform to Z transform.

- Convert Laplace transform to time domain and then to Z transform
- Use Table of z transforms to directly convert

Example

$\frac{1}{s(s+1)}$ is $1 - e^{-t}$ in time domain. Refer to the table.
$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$ and then refer to the table

Important properties of z transforms

- Scaling: if $X(z)$ is the Z transform of $x(k)$,
  $\mathcal{Z}[ax(k)] = aX(z)$.
- Linearity of the transform:
- Multiplication of a^k: if $X(z)$ is the z transform of $x(k)$,
  $\mathcal{Z}[a^k x(k)] = X(a^{-1}z)$, i.e., replace z with $a^{-1}z$. Note that k represents the time.

Two translation theorems

- Real translation theorem: If $x(t) = 0$ for $t < 0$, and if $X(z)$ is the z transform of $x(t)$,

$$\mathcal{Z}(x(t - nT)) = z^{-n}X(z) \quad \textit{backward}$$

$$\mathcal{Z}(x(t + nT)) = z^{n}(X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}) \quad \textit{forward}$$

- Complex translation theorem: if $x(t)$ has the z transform of $X(z)$,

$$\mathcal{Z}(e^{-at}x(t)) = X(ze^{aT})$$

Example

Step function

Example (discrete integrator)

Example: complex translation

Two value theorems

- Initial value theorem (IVT): If $x(t)$ has the Z transform of $X(z)$ and $\lim_{z\to\infty} X(z)$ exists, then $x(0) = \lim_{z\to\infty} X(z)$: Good for checking the correctness of a Z transform. Why?
- Final value theorem (FVT): Let the Z transform of $x(t)$ be $X(z)$. If all the poles of $X(z)$ lie inside the unit circle (one simple pole at $z = 1$ is fine), then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z)$$

- The pole condition is used to ensure that $x(k)$ remain finite.
- Poles/zeros of $X(z)$

Pole and zeros in the z plane

Given a z-transform

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}, \quad m \leq n$$

- Write it in the pole-zero form

$$X(z) = \frac{b_0 (z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)}, \quad m \leq n$$

- poles: p_i's, zeros: z_i's

• Poles/roles determine the characteristics of the $x(k)$.
• Another way of writing it using z^{-1}: convert it to the expression using z

Example (sinusoid)

Proof

The Inverse Z transform

The Z transformation serves the same role for DT control that the Laplace transform serves for CT control.

- Inverse z transform: $\mathcal{Z}^{-1}$: finding $x(kT)$ or $x(k)$ given a $X(z)$.
- Inverse z transform only gives the time sequence at the sampling instants: It gives a unique $x(k)$ but not a unique $x(t)$.
- Different $x(t)$ can have the same $x(kT)$. Later we will talk about in what cases you can recover the exact $x(t)$.
- Four methods: direct division, partial fraction expansion, inversion integral, and Matlab/computer tool.

Direct division

Expand $X(z)$ into an infinite power series of z^{-1}.

From the definition of z transform

$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} = x(0) + x(1T)z^{-1} + x(2T)z^{-2} + \cdots$

- Useful for finding the first few elements of $x(k)$ or when you can't find the closed-form solution

• Arrange the denominator and the numerator such that they are in increasing powers of z^{-1}.

• Calculate the division

Example: $X(z) = \frac{10z+5}{(z-1)(z-0.2)}$

Partial-fraction-expansion approach

- Similar techniques used in Laplace transform
- Decompose $X(z)$ into sum of terms that are easily recognizable in the table of Z transform.
- Expand $X(z)$ or $X(z)/z$, depending on whether $X(z)$ has any zeros at 0

  • make sure that the numerator is a lower order polynomial than the denominator.

  $X(z) = \frac{3z^2+2}{z^2+z+1}$

Procedures

Consider

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}, \quad m \leq n$$

- Write it in the pole-zero form

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{(z - p_1)(z - p_2) \cdots (z - p_n)}, \quad m \leq n$$

• Suppose that $b_m = 0$ and all the poles are of simple order: Expand $X(z)/z$ into

$$\frac{X(z)}{z} = \frac{a_1}{z - p_1} + \cdots + \frac{a_n}{z - p_n}$$

where

$$a_i = \left[(z - p_i) \frac{X(z)}{z} \right]_{z = p_i}$$

$$
\begin{aligned}
X(z) &= \frac{a_1 z}{z - p_1} + \cdots + \frac{a_n z}{z - p_n} \\
Z^{-1}(X(z)) &= Z^{-1}\left(\frac{a_1 z}{z - p_1}\right) + \cdots + Z^{-1}\left(\frac{a_n z}{z - p_n}\right) \\
&= \\
&=
\end{aligned}
$$

Multiple poles

Say

$$\frac{X(z)}{z} = \frac{c_1}{(z-p_1)^2} + \frac{c_2}{(z-p_1)} + \cdots$$

Then the coefficients are determined as

$$c_1 = \left[(z-p_1)^2 \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_2 = \left\{\frac{d}{dz}\left[(z-p_1)^2 \frac{X(z)}{z}\right]\right\}_{z=p_1}$$

Table of z tranforms

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform
1. $\delta[n]$	1
2. $u[n]$	$\frac{1}{1-z^{-1}}$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$
4. $\delta[n-m]$	z^{-m}
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$

Example: $X(z) = \frac{2z^3+z}{(z-2)^2(z-1)}$

Example (complex poles): $X(z) = \frac{z^3}{(z-1)(2z^2-2z+1)}$

Inversion integral method

$$\mathcal{Z}^{-1}[X(z)] = x(kT) = x(k) = \frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz, k = 0, 1, \cdots$$

where C is a circle with its center at the origin of the z plane such that all poles of $X(z)z^{k-1}$ are inside it.

- Note that for each k, the integral may be different.
- Using theory of complex variables, we have

$$\begin{aligned} x(kT) &= x(k) = K_1 + K_2 + \cdots + K_m \\ &= \sum_{i=1}^{m}[\text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}] \end{aligned}$$

  m is the number of poles (a pole with multiple orders is counted as one pole.)
- Simple pole at $z = z_i$: $K_i = \lim_{z\to z_i}(z - z_i)X(z)z^{k-1}$
- A pole z_j of order q:
  $K_j = \frac{1}{(q-1)!}\lim_{z\to z_j}\frac{d^{q-1}}{dz^{q-1}}[(z - z_j)^q X(z)z^{k-1}]$

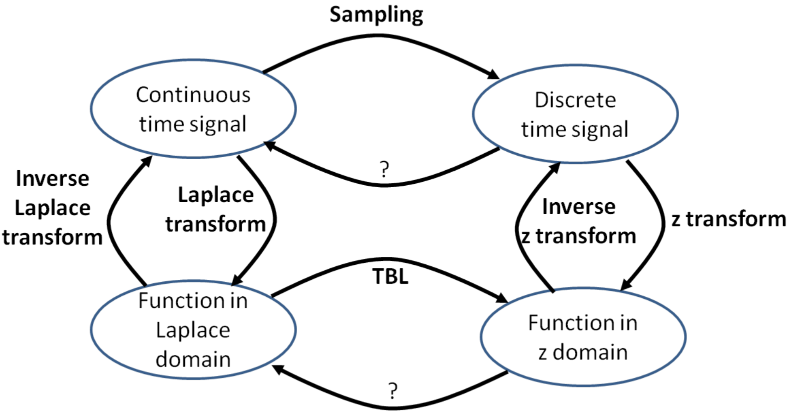
Inversion integral method

The number of poles of $X(z)z^{k-1}$ may depend on k. • $X(z)$ has one or more zeros at 0: solve for any k in closed-loop form, i.e., solve for all k.
• $X(z)$ has no zeros at 0 or has poles at 0: for different k's, you will have different poles, then you need to solve one k by another

Example: $X(z) = \frac{z^{-2}}{(1-z^{-1})^3}$

Example: $X(z) = \frac{10}{(z-1)(z-2)}$

Conversion diagram



Solving difference equation (DE)

- LTI system by linear DE:

$$x(k)+a_1x(k-1)+\cdots+a_nx(k-n) = b_0u(k)+b_1u(k-1)+\cdots+b_nu(k-n)$$

- Solve it by computer, if you know the coefficients and required values
- Z-transform allows us to calculate the closed-form solution to $x(k)$.
- Take the Z-transform and apply the shift theorem (forward/backward). Then apply inverse Z transform techniques to get to the time domain.

$$\mathcal{Z}(x(t-nT)) = z^{-n}X(z) \quad \textit{backward}$$

$$\mathcal{Z}(x(t+nT)) = z^{n}(X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}) \quad \textit{forward}$$

$$Z(x(k) + a_1 x(k-1) + \cdots + a_n x(k-n)) = X(z) + a_1 z^{-1} X(z) + \cdots a_n z^{-n} X(z$$
$$Z(b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)) =$$