# Digital Control Systems
# MAE/ECEN 5473

State space analysis

Oklahoma State University

August 14, 2023

# Modern approach: state space

- Transfer function (TF) approach

- Modern control approach: state space (SS)

# Example in continuous time

Model of a satellite heading control: $J\ddot{\theta} = \tau$

TF approach

SS approach

# Servo-motor example in SS

$$J\ddot{\theta} + b\dot{\theta} = \frac{k_T}{R_a} e(t)$$

# Example in DT

$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$$

# Classification of systems

Continuous time

- Linear time-invariant systems (LTI): $\dot{x} = Ax + Bu$, $y = Cx + Du$
- Linear time-varying systems (LTV): $\dot{x} = A(t)x + B(t)u$, $y = C(t)x + D(t)u$
- Nonlinear time-invariant systems: $\dot{x} = f(x, u)$, $y = h(x, u)$
- Nonlinear time-varying systems: $\dot{x} = f(x, u, t)$, $y = h(x, u, t)$

Discrete time

- Linear time-invariant systems (LTI): $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$
- Linear time-varying systems (LTV): $x(k+1) = A(k)x(k) + B(k)u(k)$, $y(k) = C(k)x(k) + D(k)u(k)$
- Nonlinear time-invariant systems: $x(k+1) = f(x(k), u(k))$, $y(k) = h(x(k), u(k))$
- Nonlinear time-varying systems: $x(k+1) = f(x(k), u(k), k)$, $y(k) = h(x(k), u(k), k)$

# Review of matrices & linear algebra

- Symmetric matrix, orthogonal matrix
- det, inverse, matrix operations, derivative of $A(t)$
- Linear independence of vectors $\star$
- Rank of a matrix
- Eigenvalues, eigenvectors
- Similarity transformation
- Matrix exponential

# Back to DT SS

Obtain a DT SS representation from

- a CT SS representation of a system
- a Pulse Transfer Function (PTF)

# From a CT SS representation

Given $\dot{x} = Ax + Bu$, obtain a DT SS representation with a sampling time $T$:

$$x(k+1) = F(T)x(k) + G(T)u(k)$$

- Objective: $x(kT) = x(k) = x(t)|_{t=kT}$.
- Assumption: $u$ is constant between two consecutive sampling instants.

# Derivation of the conversion formula

- $F(T) = e^{AT}$, $G(T) = \int_0^T e^{A\lambda} d\lambda B$.
- If $A$ is nonsingular,
  $G(T) = A^{-1}(e^{AT} - I)B = (e^{AT} - I)A^{-1}B$.

# Example

$$\dot{x} = -ax + u, \; x \in \mathbb{R}$$

# Example: double integrator

$\dot{y} = u$, $y \in \mathbb{R}$, $y$: position, $u$: acceleration

# DT SS representation from PTF

Given $\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 b^{n-1} + \cdots + a_{n-1} z + a_n}$, its SS representation is not unique.

- controllable canonical form (CCF)
- observable canonical form (OCF)
- Diagonal/Jordan form

# CCF

Single input single output: $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$

# Alternative CCF

# OCF

Given $(F, G, C, D)$ from CCF, OCF is obtained simply as

$$x(k+1) = F^T x(k) + C^T u(k)$$
$$y(k) = G^T x(k) + Du(k)$$

# Diagonal form

Expand $\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 b^{n-1} + \cdots + a_{n-1} z + a_n}$ into $b_0 + \sum_{i=1}^{n} \frac{c_i}{z - p_i}$

# Jordan form

$\frac{Y(z)}{U(z)}$ has a multiple pole at say $z = p_1$ of order $m$, i.e.,

$$\frac{Y(z)}{U(z)} = b_0 + \left[\frac{c_1}{(z-p_1)^m} + \frac{c_2}{(z-p_2)^{m-1}} + \cdots + \frac{c_m}{(z-p_1)}\right] + \frac{c_{m+1}}{(z-p_{m+1})} + \cdots + \frac{c_n}{(z-p_n)}$$

# Example

Write $\frac{5}{(z+1)^2(z+2)}$ into CCF, OCF and diagonal/Jordan form.

# SS representation is not unique

Suppose that we have a SS representation:

$$\begin{cases} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{cases}$$

- Define a new state $\hat{x}(k) = Px(k)$, where $P$ is invertible.

a new SS: $\begin{cases} \hat{x}(k+1) &= [?]\hat{x}(k) + [?]u(k) \\ y(k) &= [?]\hat{x}(k) + [?]u(k) \end{cases}$

- Derive expressions for [?].

# Convert SS to PTF

Apply $Z$ transform to $\begin{cases} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{cases}$

# Example: double integrator

## Solve DT SS equations, given $x(0)$ and $u(k)$

$$\begin{cases} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{cases}$$

$k = 0$:

$k = 1$:

$k = 2$:

Final formula: $x(n) =$

# Z-transform method

$$
\begin{aligned}
x(k+1) &= Gx(k) + Hu(k) \\
&\Rightarrow Z(x(k+1)) = Z(Gx(k)) + Z(Hu(k)) \\
&\Rightarrow \\
&\Rightarrow X(z) = \\
&\xrightarrow{\text{inv } Z} x(k) = Z^{-1}
\end{aligned}
$$

# Computation of $(zI - G)^{-1}$

$$(zI - G)^{-1} = \frac{adj(zI - G)}{|zI - G|}$$

Method to compute $(zI - G)^{-1}$:

Adjugate matrix

1. Determine the characteristic equation $|zI - G| = 0$ as

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

2. Compute $adj(zI - G) = Iz^{n-1} + H_1 z^{n-2} + H_2 z^{n-3} + \cdots + H_{n-1}$