Introduction to (Bayesian) Estimation
MAE 5020

Least-squares estimation

Oklahoma State University

August 14, 2023

Notation

- $\hat{\theta}(k)$:
- $\tilde{\theta}(k)$:
- $\hat{x}(k_1|N)$:
- Three situations:

- Predicted measurements:
- Prediction error:

Least-squares estimation (Batch process)

Consider N measurements: $Z(k) = H(k)\theta + V(k)$, $\theta \in \mathbb{R}^n$
In the expanded form:

$N < n$:

$N = n$:

$N > n$:

LSE solution strategy

- Define prediction error
- Find $\hat{\theta}(k)$ that minimizes

$$J(\hat{\theta}(k)) =$$

Derivation

Expand $J(\hat{\theta}(k))$ and take its derivative w.r.t. $\hat{\theta}(k)$.

Comments

- $H^T(k)W(k)H(k)$:

- 2nd derivative of $J(\hat{\theta}(k))$:

- *Linear estimator*:

Comments

- Orthogonality condition:

- Choice of $W(k)$:

- *Randomness in the estimation*:

Example: sample mean is a LSE

Example: simplified pitch dynamics

$$\ddot{\theta}(t) = M_\alpha \underbrace{\alpha(t)}_{AOA} + M_\delta \underbrace{\delta(t)}_{\text{control surface deflection}}$$

- Measurements: $\ddot{\theta}_M(k) = M_\alpha \alpha(k) + M_\delta \delta(k) + v_{\ddot{\theta}}(k)$
- LSE:

Final comments

- LS estimates may not invariant under changes of scale (units).

- Pseudo-inverse of $H(k)$

- Total LS:

Least-squares estimation (Recursive process)

- Batch process of N measurements to estimate $\theta \in \mathbb{R}^n$ is not efficient when

- Recursive estimation:

Information form: Derivation

Suppose we have a new measurement at t_{k+1} given by

$$z(k+1) = h^T(k+1)\theta + v(k+1).$$

Total measurements including previous measurements up to k

$$Z(k+1) = H(k+1)\theta + V(k+1)$$

where

$$Z(k{+}1) = \left(\qquad \right), \; H(k{+}1) = \left(\qquad \right), \; V(k{+}1) = \left(\qquad \right)$$

Objective: Write $\hat{\theta}_{WLSE}(k+1)$ in terms of $\hat{\theta}_{WLSE}(k)$ and the new measurement $z(k+1)$.

Step 1: $\hat{\theta}_{WLSE}(k+1)$

- $\hat{\theta}_{WLSE}(k+1)$ is given by

- $W(k+1)$ is a diagonal matrix, i.e.,

- Substituting $W(k+1)$ and computing $H^T(k+1)W(k+1)Z(k+1)$ as

Step 2: $\hat{\theta}_{WLSE}(k)$

- $\hat{\theta}_{WLSE}(k)$ is given by

- 1): $P(k)^{-1}\hat{\theta}_{WLSE}(k) = H^T(k)W(k)Z(k)$

- 2): $P(k+1)^{-1} = (H^T(k+1)W(k+1)H(k+1)) = P(k)^{-1} + h(k+1)w(k+1)h^T(k+1)$

Step 3: recursive form using step 1 and step 2

Observations

- $h^T(k+1)\hat{\theta}_{WLSE}(k)$ is a prediction of $z(k+1)$ based on information up to k.
- Additional form:
- Computational order of two recursions:
- Initialization:

Observations

- Matrix inversion lemma:

- On computing $P(k+1)$ using 2) for scalar case.

- Why information form?

Covariance form

Comments

- Similar to Kalman filter equations. Will disucss later.
- Computation order:
- Initilization remains the same as the information form.
- Equivalence between the covariance and the information form:
- Comparison

Example: recursive form for sample mean estimation