Introduction to (Bayesian) Estimation
MAE 5020

Elements of Gauss-Markov Random Sequences

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Overview

- Transition from parameter estimation to recursive state estimation
- First-order Markov sequence
- Gaussian White Noise
- A basic state-variable model

Random sequence and process

- Scalar random sequences and processes: scalar random sequence consists of a group of related scalar random variables at discrete points in time or space. "process" assumes continuous in time or space.

- Stationary:

- Wide-sense stationary:

Random sequence and process

- Non-stationary:
- Ergodic:

Formally

- **A vector random process** is a family of random vectors $\{s(t), t \in J\}$ indexed by t all of whose values lie in an index set J. When $J = \{k = 0, 1, \cdots\}$ (discrete), $\{s(t), t \in J\}$ is called a discrete time random **sequence**.
- **A vector random sequence** $\{s(t), t \in J\}$ is **multivariate Gaussian** if for **any** ℓ time points, $t_1, \cdots, t_\ell$, the set of ℓ random vectors $s(t_1), \cdots, s(t_\ell)$ is jointly Gaussian.
- A vector random *sequence* $\{s(t), t \in J\}$ is **a (first-order) Markov sequence**, if for any m (integer) time points $t_1 < t_2 < \cdots < t_m$, the probability law (e.g., pdf) at t_m depends only on the immediate past value at t_{m-1}.
  - For example for continuous random variables, $p[s(t_m)|s(t_{m-1}, \cdots, s(t_1)] = p[s(t_m)|s(t_{m-1})]$.
  - If $s(t_m)$ depends on both $s(t_{m-1})$ and $s(t_{m-2})$, it is called *second-order Markov sequence*.

Fact

Let $s(t)$ be a first-order Markov sequence. Suppose $t_1 < t_2 < \cdots < t_m$.

$$p[s(t_m), s(t_{m-1}), \cdots, s(t_1)] =$$

$$p[s(t_m)|s(t_{m-1}]p[s(t_{m-1})|s(t_{m-2}] \cdots p[s(t_2)|s(t_1)]p[s(t_1)].$$

Comments

- A Markov sequence depends on

- Easy to show $E[s(t_m)|s(t_{m-1}), \cdots, s(t_1)] =$

Gaussian white noise

A vector random sequence $s(t)$ is a Gaussian white noise/sequence if for any m (integer) time points $t_1 < t_2 < \cdots < t_m$, the random vectors $s(t_1), \cdots, s(t_m)$ are uncorrelated Gaussian random vectors.

- Properties:

The basic state-variable model: linear time-varying

$$
\begin{aligned}
x(k+1) &= \Phi(k+1,k)x(k) + \Gamma(k+1,k)w(k) + \Psi(k+1,k)u(k) \\
z(k+1) &= H(k+1)x(k+1) + v(k+1), \quad k = 0, 1, \cdots .
\end{aligned}
$$

Comments

- Φ, Γ and Ψ may not always depend on $(k+1)$ or k.
- $w(k)$ is used to model uncertainties in the process, including
- $v(k)$:
- This is a basic linear model.

Properties

When $x(0)$ and $w(k)$ are jointly Gaussian, $x(k)$, $k = 0, 1, \cdots$ is a Gauss-Markov sequence.

Properties

A Gauss-Markov sequence can be characterized by the mean and the covariance of the state vector sequence.

Propagation of Mean: $m_x(k) = \mathbb{E}(x(k))$

$$m_x(k+1) = \Phi(k+1,k)m_x(k) + \Psi(k+1,k)u(k), \quad k = 0, 1, \cdots .$$

Propagation of covariance

Recall $P_x(k) = \mathbb{E}(x(k) - m_x(k))(x(k) - m_x(k))^T$. Then,

$$P_x(k+1) = \Phi(k+1, k)P_x(k)\Phi^T(k+1, k)+\Gamma(k+1, k)Q(k)\Gamma^T(k+1, k)$$

Cross-covariance and measurements

Recall $E[(x(i) - m_x(k))(x(j) - m_k(j))^T] = P_x(i,j)$. Then

$$P_x(i,j) = \begin{cases} \Phi(i,j)P_x(j) & i > j \\ P_x(i)\Phi^T(j,i) & i < j. \end{cases}$$

Measurement model for $z(k+1)$ is also Gaussian with

$$m_z(k+1) = H(k+1)m_x(k+1) \quad \because E(v(k)) = 0$$
$$P_z(k+1) = H(k+1)P_x(k+1)H(k+1)^T + R(k+1).$$

Example

$$x(k+1) = 1/2x(k) + w(k)$$
$$z(k+1) = x(k+1) + v(k+1)$$

$m_x(0) = 4$, $p_x(0) = 10$, $q = 20$, and $r = 5$.

Stationary sequence

- Stationary means:

- If in addition, Φ is asymptotically stable (all poles lie inside the unit circle), $P_x(k)$ converges to a limiting solution $\bar{P}_x$ satisfying the *discrete time Lyapunov equation*

$$\bar{P}_x = \Phi \bar{P}_x \Phi^T + \Gamma Q \Gamma^T.$$