Digital Control Systems
MAE/ECEN 5473

Control design in frequency domain

Oklahoma State University

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What we have covered so far:

- Conventional (classical): Analysis Design
- Modern (state space): Analysis Design

Conventional control design

- Based on Z-transform
- Open-loop system vs. closed-loop system in block diagrams

- Objective: design $G_D(z)$ such that the closed-loop system satisfies certain performance requirements.

Performance specifications

- Transient performance
- Steady-state performance
- Standard testing signal for r (input):
  - (unit) step function
  - ramp function
  - acceleration function
  - sinusoidal funcation

Transient performance

Given a step input (assume all initial conditions are 0):

- delay time (t_d): time when the output reaches $\frac{C_F}{2}$
- rise time (t_r): time the output takes to rise from 10% C_F to 90% C_F
- peak time (t_p): time when the output reaches the first peak
- Maximum shoot: the output at t_p, $y(t_p)$
- Overshoot percentage: $\frac{y(t_p)-C_F}{C_F} \times 100\%$
- Settling time (t_s): time required for the output to settle within 2% (5%) of the final value C_F

Map damping ratio/natural frequency in Z-domain

Recall $z = e^{Ts}$. Since $s = \sigma + j\omega$,

$$z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega}. \quad (1)$$

Consider the 2nd order system: $G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$

- Poles: $s_{1,2} = -\zeta w_n \pm \sqrt{1 - \zeta^2} w_n j$ w_n, ζ?
- From (1), $z_{1,2} = e^{Ts_{1,2}} = e^{-\zeta w_n T} e^{\pm T\sqrt{1-\zeta^2} w_n j}$.
- Thus, $|z_{1,2}| = e^{-\zeta w_n T}$, $\angle z_{1,2} = \pm T\sqrt{1 - \zeta^2} w_n$.
- Given ζ, w_n, we can find the corresponding poles in DT: $z_{1,2} = e^{-\zeta w_n T} e^{\pm T\sqrt{1-\zeta^2} w_n j}$.

Quiz

Given a z-domain pole $z = r\angle\theta$ (complex poles), find the corresponding ζ, w_n. Also compute the time constant $\tau = \frac{1}{\zeta w_n}$ in terms of T, r, θ.
Hint: Start with $r = e^{-\zeta w_n T}$ and $\theta = T\sqrt{1-\zeta^2}w_n$ to solve for w_n and ζ in terms of r, θ, T.

Steady-state (SS) performance

- Measured by SS errors (use Final value theorem (FVT), **assume FVT can be applied**)
- Type 0, I, II, ... systems:
  - Type 0: finite SS error for step input, infinite error for higher order inputs (ramp, acceleration, ...)
  - Type I: 0 SS error for step input, finite SS error for ramp input, infinite SS error for higher order input (acceleration, ...)

input signal	step	ramp	acceleration	higher order
Type 0	finite error	∞	∞	∞
Type I	0	finite error	∞	∞
Type II	0	0	finite error	∞

As the system type increases

- better SS performance
- difficult to stabilize

System type

- Suppose that the open-loop PFT is $\frac{1}{(z-1)^N}\frac{B(z)}{A(z)}$, where $B(z)$ and $A(z)$ contain no zero or pole at $z = 1$. The closed-loop system is type N, provided that the closed-loop system is stable.
- Example for $N = 1, 2$ with the block diagram

Root locus based control design

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where $G(z) = Z(G(s))$ and $GH(z) = Z(G(s)H(s))$.

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The characteristic equation of the PTF is given by

$$\star : 1 + KGH(z) = 0, \quad KGH(z) = \frac{num(z)}{den(z)}. \tag{2}$$

Root locus (RL) is the plot of the locus of the roots of $\star$ in the z-plane as a function of K.

- The construction procedure of RL in DT is the same as in CT.
- The stability region in DT is the unit circle while in CT it is the open left half plane.

RL drawing rules (only important ones)

1. RL originates from the poles of $GH(z) = 0$ ($den(z) = 0$) and terminate at the zeros of $GH(z) = 0$ ($num(z) = 0$) (including ∞). why?

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$$1 + KGH(z) = 0 \Rightarrow 1 + K\frac{num(z)}{den(z)} = 0 \Rightarrow den(z) + Knum(z) = 0$$

RL starts at $K = 0 \Rightarrow den(z) = 0 \Rightarrow$ poles of $GH(z)$
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3. The RL is symmetric w.r.t. the real axis.

Continued: RL drawing rules (only important ones)

4. The number of asymptotes of RL equals $n_p - n_z$, n_p, n_z are the number of the poles and zeros of $GH(z)$, respectively. The angles of the asymptotes are $\frac{(2k+1)\pi}{n_p - n_z}$, $k = 0, 1, \cdots$.
5. The asymptotes intersect the real axis at $\sigma = \frac{\sum \text{poles of } GH(z) - \sum \text{zeros of } GH(z)}{n_p - n_z}$
6. The breakaway points are given by the roots of $\frac{dGH(z)}{dz} = 0$.

Example: $KGH(z) = \frac{K0.368(z+0.717)}{(z-1)(z-0.368)}$

Design via RL

Principle

Consider a closed-loop system and its characteristic equation $1 + KD(z)G(z) = 0$. Then a point z_a is on the RL if $1 + KD(z_a)G(z_a) = 0$ for some K and $D(z)$.

$$1 + KD(z_a)G(z_a) = 0 \Leftrightarrow \begin{cases} K = \frac{1}{|D(z_a)G(z_a)|} \\ \angle D(z_a)G(z_a) = \pm 180^\circ \end{cases} \tag{3}$$

Design $D(z)$ such that the closed-loop poles satisfy certain requirements, such as a desired damping ratio and settling time.

Example (MATLAB code): $G_p(s) = \frac{1}{s(s+2)}$

Control design based on frequency response

In CT, given a stable LTI system $G(s)$ with $\sin(wt)$ as input:

$$x_{ss}(t) = \underbrace{|G(j\omega)|}_{magnitude} \sin(wt + \underbrace{\angle G(j\omega)}_{phase\ \ shift})$$

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In DT, given a stable LTI system $G(z)$ with **sampled** $\sin(wt)$ as input:

$$x_{ss}(kT) = \underbrace{|G(e^{j\omega T})|}_{magnitude} \sin(kwt + \underbrace{\angle G(e^{j\omega T})}_{phase\ \ shift})$$

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Observation

$G(z)|_{z=e^{j\omega T}}$ provides magnitude and phase information of the frequency response of $G(z)$.

w-plane

- Evaluating $G(e^{j\omega T})$ can be complex.
- Simplify the procedure by transforming the design from z-domain to w-domain via the "bilinear" transformation

$$w = \frac{2}{T}\frac{z-1}{z+1} \Leftrightarrow z = \frac{1+\frac{T}{2}w}{1-\frac{T}{2}w}$$

- w-plane and s-plane are different. Why?

Design in the w-plane

Once $G(z)$ is transformed to $G(w) = G(z)\big|_{z=\frac{1+\frac{T}{2}w}{1-\frac{T}{2}w}}$, it may be treated as a conventional TF in w-domian. Conventional frequency response based designs can be applie to $G(w)$.

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When ωT is small,

$$\nu = \frac{2}{T}\tan\frac{\omega T}{2} \approx \frac{2}{T}\frac{\omega T}{2} = \omega$$

ν : ficticious frequency in w-plane, ω: true frequency

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Example

If the design requirement says a bandwidth ω_b, the corresponding bandwidth in the w-plane is $\nu_b = \frac{2}{T}\tan\frac{\omega_b T}{2}$.

Phase lead/lag design (design in w-plane)

1. Obtain $G(z)$ and transform it to $G(w) = G(z)\big|_{z=\frac{1+\frac{T}{2}w}{1-\frac{T}{2}w}}$.
2. Substitute $j\nu$ as w in $G(w)$ and obtain the Bode plot $|G(j\nu)|$ and $\angle G(j\nu)$.
3. Use the conventional design methods for CT systems to determine $G_D(w)$ such that the open-loop system $G_D(w)G(w)$ satsifies design specifications (e.g., phase margin, gain margin, DC gain condition)
4. Transform $G_D(w)$ to $G_D(z) = G_D(w)\big|_{w=\frac{2}{T}\frac{z-1}{z+1}}$.
5. Realize $G_D(z)$ via a computaional algorithm, e.g., convert it to a difference equaiton or a SS representation

$\star$: Frequency axis in the w-plane is distorted. what is the relationship?

Review of Bode plot in CT

- Definition
- how to sketch
- Gain and phase margins
- Example

Phase lag compensation in DT

Compensator: $G_D(w) = k_D \frac{1+\tau w}{1+\beta\tau w}$, $(k_D > 0,\ \tau > 0,\ \beta > 1)$

- Pole:
- Zero:
- Bode plot for $G_D(w)$
- Magnitude and phase plots

Closed-loop system diagram

Open-loop system $G_D(w)G(w) = k_D \frac{1+\tau w}{1+\beta\tau w} G(w)$.

Determine the compensator

- Determine k_D based on gain conditions:

- Design τ and β to yield a desired phase margin ϕ_m (given). Let $G_1(w) = k_D G(w)$ (known).
  - Create the bode plot of $G_1(w)$ (magnitude and phase)
  - Determine the frequency w_{w_1} at which the phase plot of $G_1(j\nu)$ has a phase angle of $-180^\circ + \phi_m + 5^\circ \sim 12^\circ$ (to compensate for the phase lag of $G_D(w)$)
  - w_{w_1} is the frequency at which the phase margin occurs (after compensation).
  - Set $\frac{1}{\tau} = 0.1 w_{w_1}$ (far away from w_{w_1})
  - At w_{w_1}, we would like $|G_D(jw_{w_1})G_1(jw_{w_1})| = 1$, which leads to

Convert to z-domain

- Once $G_D(w)$ is solved from the previous procedure,

$$G_D(z) = G_D(w)|_{w=\frac{2}{T}\frac{z-1}{z+1}} = k_D \frac{(1+\frac{2\tau}{T})z + 1 - \frac{2\tau}{T}}{(1+\frac{2\beta\tau}{T})z + 1 - \frac{2\beta\tau}{T}}. \quad (4)$$

Phase lead compensators in DT

Compensator: $G_D(w) = k_D \frac{1+\tau w}{1+\alpha\tau w}$, $(k_D > 0,\ \tau > 0,\ \alpha > 1)$

- Pole:
- Zero:
- Bode plot for $G_D(w)$
- Magnitude and phase plots

Open-loop TF: $G_D(w)G(w) = k_D \frac{1+\tau w}{1+\alpha\tau w} G(w) = \frac{1+\tau w}{1+\alpha\tau w} \underbrace{[k_D G(w)]}_{G_1(w)}$

Design procedure

- Determine k_D based on a given gain or static velocity error constant.
- Create bode plots of $G_1(w)$ and evaluate the phase margin.
- Determine the necessary phase lead angle ϕ to be added to the system (based on the phase margin requirement)
- Add $5 \sim 12^\circ$ to ϕ to compensate for the shift of the crossover frequency
- $\phi + 5 \sim 12^\circ = \phi_m$. From $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$, solve α.
- Select $v_m = \frac{1}{\sqrt{\alpha\tau}}$ as the gain crossover frequency. How?
- Set $\tau = \frac{1}{\sqrt{\alpha v_m}}$.
- Check the phase margin and repeat the process by modifying pole/zero locations.
- Convert $G_D(w)$ into z-domain using $w = \frac{2}{T}\frac{z-1}{z+1}$.

Example (MATLAB code)