Digital Control Systems
MAE/ECEN 5473

Modeling of Sample and Hold Process

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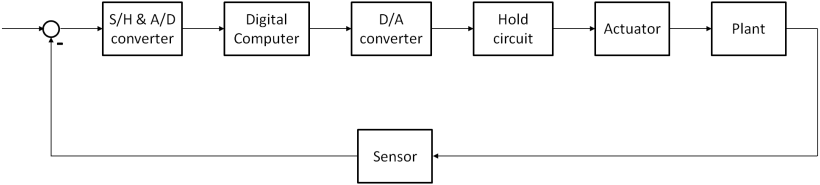
Last time

- z transforms and inverse z transforms
- Using z transforms to solve difference equations

review of the diagram

car driving problem with the diagram

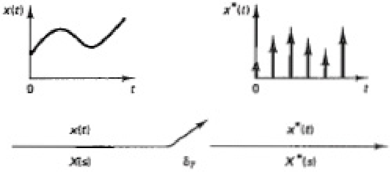


This new chapter

- Objective: analyze the effect of the sampler and hold using z transform as a tool
- Assumptions: single rate, synchronized sampling if multiple samplers are used
- First, develop models for impulse sampling and data hold and thus the interface is modeled (show the relationship to the z transform)
- Second, combine it with the plant and obtain the total transfer function
- Also study more about sampling: sampling frequency requirements, others such as aliasing, folding phenomenon

Impulse sampling: Model the sampler

- Impulse sampler: Fictitious sampler, output is considered a train of impulses beginning at $t = 0$ with sampling period T. The magnitude at each pulse is the sampled value of the continuous-time signal at each sampling instant.



- $\delta(t)$ function: unit impulse function, $\delta(t) = 1$ if $t = 0$, otherwise, $\delta(t) = 0$.

Output of sampler

- The impulse-sampled output $x^{\star}(t)$ of $x(t)$ is a sequence of impulses as shown above:

$$x^{\star}(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT),$$

or (star-transform)

$$x^{\star}(t) = x(0)\delta(t) + x(T)\delta(t - T) + x(2T)\delta(t - 2T) + \cdots$$

- Define a train of unit impulses as $\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$. Then $x^{*}(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT) =$

Laplace transform of the output

$$X^\star(s) = \mathcal{L}[x^\star(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t-T)] + \cdots$$

- $\mathcal{L}[\delta(t-nT)] = e^{-nTs}$, $n = 0, 1, \cdots$
- $X^\star(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots$
- Exactly the same as Z transform if we say $e^{Ts} = z$.
- So $X^\star(s)|_{s=1/T \ln z} = X(z)$. That is: the Laplace transform of the output of the impulse sampler is equivalent to the Z-transform of the output if $e^{Ts} = z$.
- Impulse sampling used as a fictitious sampler and does NOT exist in practice.

From Laplace transform (s-domain) to Z-transform (z-domain)

Goal: Convert $X(s) = \mathcal{L}(x(t))$ to $X(z)$ (and $X^*(s)$)

Recall $\mathcal{L}[f(t)g(t)] =$

Given $\mathcal{L}[x(t)] = X(s)$ and $\mathcal{L}[\sum_{k=0}^{\infty} \delta(t - kT)] =$
we compute $X^*(s) = \mathcal{L}(x^*(t)) = \mathcal{L}[x(t) \sum_{k=0}^{\infty} \delta(t - kT)] =$

Final formula: from star transform to Z-transform

Example

Modeling the data hold

Data hold is a process of generating a continuous time signal $h(t)$ based on a DT sequence $x(kT)$.

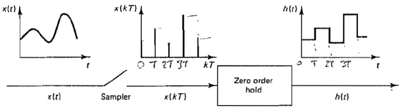
- There are different ways of holding the value or generating $h(t)$, e.g., zero order, first order, etc
- The signal $h(t)$ between kT and $(k+1)T$ may be of the form

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0, \quad 0 \le \tau < T$$

- Since $h(kT) = x(kT)$, we have $a_0 = x(kT)$.
- Given different n, we have different order of data hold: zero-order hold $(n=0)$, first-order hold $(n=1)$, ...

Zero-order hold (our assumption throughout the course)

$h(kT + \tau) = x(kT)$



- How to obtain a transfer function of the data hold: input $x^{\star}(s)$ and output $h(s)$, the laplace transform of $h(t)$.

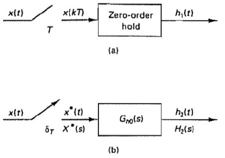


Figure 3–4 (a) A real sampler and zero-order hold; (b) mathematical model that consists of an impulse sampler and transfer function $G_{h0}(s)$

Next slide develops the transfer function. First consider $h_1(t)$ and then $h_2(t) = h_1(t)$

Development of the transfer function

First consider $h_1(t)$ in figure (a). Recall step function: $1(t-t_1)=1$, if $t \geq t_1$, otherwise, it is zero.

- $h_1(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + x(2T)[1(t-2T) - 1(t-3T)] + \cdots$
- Further obtain $h_1(t) = \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)]$
- Because $\mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$ (from laplace transform table)

$$\mathcal{L}[h_1(t)] = H_1(s) = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$\mathcal{L}[h_1(t)] = \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

TF

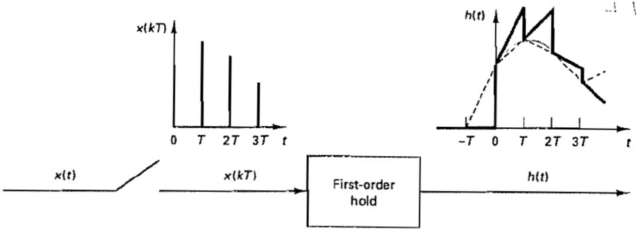
Since $h_1(t) = h_2(t)$, $H_1(s) = H_2(s) = \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs}$.

- Note that the summation term is indeed $X^{\star}(s)$, if you recall.
- So the transfer function $G_{h0}(s) = \frac{1-e^{-Ts}}{s}$.

Note that the two figures are mathematically equivalent from the input-output relationship, i.e., *a real sampler and zero order hold can be replaced by a (mathematically) equivalent continuous-time system that consists of an impulse sampler and a transfer function* $\frac{1-e^{-Ts}}{s}$.

First order hold

- First order hold: $G_{h1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}$. We will omit the derivation.



How the extrapolation is done in the figure?

Recap

- Impulse sampler model: output is the star transform of the signal (z-transform with z replaced by e^{Ts}.)
- Hold process: transfer function given by $(1 - e^{-Ts})/s$.
- Difficulty to work with e^{Ts}. How can we make use of z transform?
- Impulse sampler can be easily converted to z transform
- Hold process is usually followed by a continuous process to be controlled $G(s)$
- If we can actually convert $(1 - e^{-Ts})/s \times G(s)$ to z domain, then we end up with everything in z domain (note that control algorithms described by DEs can also be converted to z transforms.).

Conversion from s-domain to z-domain: background

Fact (Convolution): Laplace transform of product of two Laplace-transformable functions $f(t)$ and $g(t)$

$$\mathcal{L}[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}dt = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)dp.$$

Derivation provided in book

- For $x^{\star}(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = x(t)\sum_{k=0}^{\infty}\delta(t-kT)$, we have

$$X^{\star}(s) = \mathcal{L}[x^{\star}(t)] = \mathcal{L}[x(t)\sum_{k=0}^{\infty}\delta(t-kT)]$$

- Note that $\mathcal{L}[\sum_{k=0}^{\infty}\delta(t-kT)] = 1 + e^{-Ts} + \cdots = \frac{1}{1-e^{-Ts}}$
- Applying the above fact of Laplace transform

$$X^{\star}(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

$$X^\star(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

- shows how to convert from $X(s)$ to z domain
- How to evaluate the integral?
- What is the integral path $c - j\infty$ to $c + j\infty$? It needs to separate the poles of $X(p)$ and those of $\frac{1}{1-e^{-T(s-p)}}$.
- How to do the integral? Evaluating residues by forming a closed contour consisting of the line $c - j\infty$ to $c + j\infty$ and a semicircle Γ of infinite radius in the left or right half plane, provided that the integral along the added semicircle is constant or zero. Draw the figure (Figure 3-8 in the book)

$$X^\star(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_\Gamma X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

Evaluating with the semicircle Γ in the left half plane

$$X^\star(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_\Gamma X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

Assumptions of $X(s)$: If $X(s) = q(s)/p(s)$ with poles in the left half-plane (including imaginary axis) and $p(s)$ is of a higher order degree in s than $q(s)$, $\lim_{s\to\infty} X(s) = 0$.
Consequence Integral along Γ vanishes.
Now go from

$$X^\star(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

to Z-transform

$$X(z) = \frac{1}{2\pi j}\oint X(p)\frac{z}{z-e^{Tp}}dp$$

Evaluation

$$X(z) = \frac{1}{2\pi j} \oint X(p) \frac{z}{z - e^{Tp}} dp$$

equals $\sum$[the residue of $X(p)\frac{z}{z-e^{Tp}}$ at pole of $X(p)$] Or you replace p by s.

- simple pole: $K_j = \lim_{s \to s_j} [(s - s_j) X(s) \frac{s}{s-e^{Ts}}]$
- multiple pole of order n_i:
  $K_j = \frac{1}{(n_i-1)!} \lim_{s \to s_j} \frac{d^{n_i-1}}{ds^{n_i-1}} [(s - s_j)^{n_i} X(s) \frac{s}{s-e^{Ts}}]$
- where did we use this "residue" concept?

Example: refer to paper note

Evaluating with the semicircle Γ in the right half plane

Reserved for later: This is not useful for converting to z transform, but more related to sampling theorem

Next time

Sampling theorem

Impulse transfer function

- CT: TF relates the input and the output.
- DT: Impulse transfer function does the same thing.