Digital Control Systems
MAE/ECEN 5473

Quadratic Optimal Control

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Optimal control

- Optimal control: Design a control that yields "best" performance with respect to some "objective function"
- Cost function (performance index):

$$J_N = \sum_{k=0}^{N} L_k(y(k), x(k), u(k), r(k)) \tag{1}$$

  Design $u(k)$, $k = 0, \cdots, N$ such that J_N is minimized subject to

$$x(k+1) = Gx(k) + Hu(k)$$

- If J_N is to be maximized, then minimize $-J_N$.

Quadratic cost

$$J_N = \sum_{k=0}^{N} x(k)^T Q(k) x(k) + u(k)^T R(k) u(k) \qquad (2)$$

Design parameters: $Q(k)$ – Positive semidefinite (PSD), $R(k)$ – Positive definite (PD)

Example

$Q(k) = C^T C$.

Example

Choose $R(k)$ to be PD.

Optimal control problems

- Given $x(k+1) = Gx(k) + Hu(k)$ and $y(k) = Cx(k)$, assume full state feedback, i.e., $x(k)$ is available.
- Determine a control $u(k) = f(x(k))$ such that the cost function $J_N = \sum_{k=0}^{N} x(k)^T Q(k) x(k) + u(k)^T R(k) u(k)$ is minimized, where $Q(k)$ is PSD and $R(k)$ is PD.
- An optimal control is optimal only for the selected performance index (cost function), i.e., may not be optimal for other cost functions.

Disclaimer

In the book,
$J_N = \frac{1}{2}x(N)^T Sx(N) + \frac{1}{2}\sum_{k=0}^{N-1}(x(k)^T Qx(k) + u(k)^T Ru(k))$.

This is equivalent to our notation
$J_N = \sum_{k=0}^{N} x(k)^T Q(k)x(k) + u(k)^T R(k)u(k)$ with $Q(N) = \frac{1}{2}S$, $R(k) = \frac{1}{2}R$, $Q(k) = \frac{1}{2}Q$, $u(N) = 0$.

Quiz: Why can we set $u(N) = 0$ in our formulation?

Solve optimal control

Principle of optimality (Richard Bellman)

If the control $u(k) = f(x(k))$ is optimal over the period $0 \leq k \leq N$, then it is also optimal over any horizon $m \leq k \leq N$, where $0 \leq m \leq N$.

- Optimal control can be solved backwards in time.

PoO applied to optimal control

- Define $F(k) = x(k)^T Q(k)x(k) + u(k)^T R(k)u(k)$.
- Define S_m to be the cost from $k = N - m + 1$ to $k = N$ (cost to go), i.e., $S_m = F(N - m + 1) + F(N - m + 2) + \cdots + F(N)$.
- Note $J_N = \sum_{k=0}^{N} F(k)$ and $S_m = J_N - J_{N-m}$.

Principle

If J_N is optimized, S_m must be optimized, $m = 1, \cdots, N + 1$.

1. Start with minimizing $S_1 = J_N - J_{N-1} = F(N)$ w.r.t. $u(k)$.
2. Optimize $S_2 = F(N - 1) + F(N) = F(N - 1) + S_1^*$ where S_1^* is the optimized value of S_1.
3. $\vdots$
4. Continue this process until $S_{N+1} = J_N$ is minimized.

Example

Example continued

Observations

- The optimal control is linear: $u(k) = -K(k)x(k)$.
- The optimal control is time-varying because $K(k)$ depends on K.
- The optimal control is solved backwards in time.
- The minimum of the cost function depends on the initial condition $x(0)$. Show.

The minimum principle

Given $x(k+1) = Gx(k) + Hu(k)$, determine $u(k) = f(x(k))$ such that $J_N = \sum_{k=0}^{N} x(k)^T Q(k)x(k) + u(k)^T R(k)u(k)$ is minimized, where $Q(k)$ is PSD and $R(k)$ is PD.

Principle

If $u^*(k)$ is optimal and the corresponding $x^*(k)$ is optimal, then there exists a sequence of nontrivial vectors $\{p^*(k)\}$ such that $u^*(k)$ is the value of $u(k)$ that minimizes the Hamiltonian

$$H_k = \frac{1}{2}(x^*(k)^T Q(k)x^*(k) + u(k)^T R(k)u(k)) \\ + p^*(k+1)^T (Gx^*(k) + Hu(k)). \tag{3}$$

- $u^*(k)$ satisfies $\frac{\partial H_k}{\partial u(k)} = 0$, i.e., $u^*(k) = -R^{-1}(k)H^T p^*(k+1)$.
- $p^*(k)$ satisfies $p^*(k) = \frac{\partial H_k}{\partial x^*(k)}$, i.e., $p^*(k) = Q(k)x^*(k) + G^T p^*(k+1)$, with $p^*(N) = Q(N)x^*(N)$.

Linear two-point boundary value problem

- For $p^*(k)$, given $p^*(N)$, it is solved backward in time.
- For $x^*(k)$, given $x(0)$, it is solved forward in time.

optimal control algorithm: the optimal control is given by

$$u^*(k) = -K(k)x(k)$$

$$K(k) = [H^T P(k+1)H + R(k)]^{-1} H^T P(k+1)G \quad (\star)$$

$$P(k) = G^T P(k+1)[G - HK(k)] + Q(k) \quad (\star\star)$$

1. Start with $P(N) = Q(N), \quad K(N) = 0$.
2. Solve for $K(N-1)$ from $(\star)$
3. Solve for $P(N-1)$ from $(\star\star)$ using $P(N)$ and $K(N-1)$
4. Solve for $K(N-2)$ from $(\star)$ using $P(N-1)$ from 3.
5. Solve for $P(N-2)$ from $(\star\star)$ using $P(N-1)$ and $K(N-2)$ from 4.
6. $\vdots$
7. Solve for $K(0)$ and then $P(0)$.

Final comments

- The optimal cost J achieved by $u^*(k) = -K(k)x(k)$ is $J^* = \min J = x(0)^T P(0) x(0)$.
- So far, only considered finite horizon problems, i.e., N is finite.
- Infinite horizon: $N \to \infty$, $Q(k) = Q$, $R(k) = R$.
  - $P(k)$ in $(\star\star)$ converges to a constant P.
  - $u^*(k) = -Kx(k)$ where K is a constant gain matrix. From $(\star)$ and $(\star\star)$,

    $$K = [H^T P H + R]^{-1} H^T P G, P = G^T P [G - HK] + Q \Rightarrow$$

    Algebraic Riccati Equation (ARE):

    $$P = G^T P G + Q - G^T P H (H^T P H + R)^{-1} H^T P G$$

  - *Main advantage*: If (G, H) is controllable, P from the ARE is PD and $u(k) = -Kx(k)$ is stabilizing.
  - If the full state $x(k)$ is not available, design an observer based on $y(k) = Cx(k)$ and $u(k)$ and use the estimate $\tilde{x}(k)$ in the feedback control $u(k) = -K\tilde{x}(k)$.

Example revisited

A "very practical" introduction to Kalman filter (KF)

- Motivation
  - Reconstruct state from measurements & inputs (similar to an observer)
  - Reduce the noise impact in the estimate ("filtering")
- Setup
  - Dynamical system: $x_k = F_k x_{k-1} + G_k u_k + B_k w_k$
  - Measurement: $z_k = H_k x_k + v_k$ (observation model)
  - x_k, F_k, u_k, w_k, v_k:
- Objective: Given the noisy measurements z_k and input u_k, reconstruct x_k

KF equations

- Assumption: $\{x_0, w_1, w_2, \cdots, v_1, v_2, \cdots\}$ are mutually independent.
- Notation. $\hat{x}_{n|m}$: estimate of x at time step n (x_n) given observations up to time step m (i.e., $\{z_1, \cdots, z_m\}$). What is $\hat{x}_{k|k}$?
  $P_{k|k}$: error covariance of $\hat{x}_{k|k}$, measure of estimation accuracy
- $\mathcal{N}(\mu, \Sigma)$: Normal distribution with mean $\mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}_{>0}$ (PD matrix)

KF algorithm

KF equations: start with $\hat{x}_{0|0} \sim \mathcal{N}(x_0, P_{0|0})$
At time step k, $k \geq 1$, recursively implement

1. Prediction step (no measurements involved):

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + G_k u_k$$
$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + B_k Q B_k^T$$

2. Update step (incorporate measurements):

$$\tilde{y}_k = \underbrace{z_k}_{measured} - \underbrace{H_k \hat{x}_{k|k-1}}_{predicted}$$
$$S_k = H_k P_{k|k-1} H_k^T + R, K_k = P_{k|k-1} H_k^T S_k^{-1},$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$
$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \leq P_{k|k-1}?$$

Conclusions/Results (without proof)

- If $\hat{x}_{0|0}, P_{0|0}$ accurately reflect the true distribution of x_0, then

- KF is the optimal linear filter and minimizes the trace of $P_{k|k}$, if a) the model perfectly matches the real system, b) the noise is uncorrelated, c) R and Q are known exactly.
- If the noise (w_k, v_k) are not normal distribution, KF is still the best linear filter.
- If the dynamical system (F_k, H_k) is observable, the estimation error $\mathbb{E}(x_k - \hat{x}_{k|k})$ remains bounded.

Connection to current observer: Remove the prediction step

Example (MATLAB)